Available online at www.sciencedirect.com
SCIENCE DIRECT

Journal of APPLIED MATHEMATICS AND MECHANICS

ELSEVIER

www.elsevier.com/locate/jappmathmech

Journal of Applied Mathematics and Mechanics 68 (2004) 941–946

THE NUMBER OF INDEPENDENT COMPATIBILITY EQUATIONS IN THE MECHANICS OF DEFORMABLE SOLIDS†

D. V. GEORGIYEVSKII and B. Ye. POBEDRYA

Moscow

e-mail: georgiev@mech.math.msu.su; pobedria@mech.math.msu.su

(*Received* 30 *September* 2003)

The number of independent compatibility equations in terms of stresses, involved in formulating the basic problem in the mechanics of deformable solids in terms of stresses in $\mathbb{R}^n$, is the same as the number of Saint-Venant compatibility equations in $\mathbb{R}^n$ and the number of independent components of the Kröner and Riemann–Christoffel tensors. The existence of the Bianchi identities does not reduce this number. Counterexamples are given to show that the number of Beltrami–Mitchell equations cannot be reduced from six to three in the classical and new formulations of the problem in terms of stresses for a three-dimensional body.

1. THE NUMBER OF INDEPENDENT SAINT-VENANT COMPATIBILITY EQUATIONS

In order to represent the Saint-Venant compatibility equations [1] in compact form, one usually considers the Kröner incompatibility tensor [2]. In $\mathbb{R}^n$ this tensor is the object $\boldsymbol{\eta}^{\{2n-4\}} = \mathrm{Ink}\boldsymbol{\varepsilon}^{\{2\}}$ of rank $2n-4$ with Cartesian components

$$\eta_{i_1\ldots i_{n-2}j_1\ldots j_{n-2}} = \epsilon_{i_1\ldots i_{n-2}kl}\epsilon_{j_1\ldots j_{n-2}mp}\varepsilon_{lm,kp} \tag{1.1}$$

where $\epsilon_{i_1\ldots i_n}$ is the Levi–Civita symbol in $\mathbb{R}^n$ and $\boldsymbol{\varepsilon}^{\{2\}}$ is the strain tensor. Indeed, the vanishing of all components $\eta_{i_1\ldots i_{n-2}j_1\ldots j_{n-2}}$ is a necessary and, for a simply-connected domain V, also a sufficient condition for the Cauchy problem

$$u_{i,j}+u_{j,i} = 2\varepsilon_{ij}, \quad i,j = 1,\ldots,n \tag{1.2}$$

to be integrable. Thus, the number N_n of independent components (1.1) is the same as the required number of independent equations of strain compatibility.

For $n = 2$ the Kröner tensor is a scalar ($N_2 = 1$) and for $n = 3$ it is a symmetric rank 2 tensor ($N_3 = 6$). It is obvious from definition (1.1) that, for any n, the tensor $\boldsymbol{\eta}^{\{2n-4\}}$ is anti-symmetric in all pairs of its first $n-2$ indices, and also of the last $n-2$. Multiplying both sides of (1.1) by $\epsilon_{i_1\ldots i_{n-2}qs}\epsilon_{j_1\ldots j_{n-2}tr}$ and summing over the $2n-4$ repeated indices, one readily obtains the following relations, equivalent to (1.1)

$$2R_{sqtr} \equiv \varepsilon_{st,qr}+\varepsilon_{qr,st}-\varepsilon_{sr,qt}-\varepsilon_{qt,sr} = 0 \tag{1.3}$$

where $\mathbf{R}^{\{4\}}$ is the curvature tensor or the Riemann–Christoffel tensor, whose rank is 4 for any n. Its components admit of the classical symmetries

$$R_{sqtr} = -R_{qstr} = -R_{sqrt} = R_{trsq} \tag{1.4}$$

†*Prikl. Mat. Mekh.* Vol. 68, No. 6, pp. 1043–1048, 2004.

0021–8928/$—see front matter.
doi: 10.1016/j.jappmathmech.2004.11.015

and the Ricci identities

$$R_{sqtr} + R_{strq} + R_{srqt} = 0 \tag{1.5}$$

Thus, the incompatibility tensor $\eta^{\{2n-4\}}$ and the curvature tensor $\mathbf{R}^{\{4\}}$ are dual tensors, and consequently have the same number of independent components. The geometrical meaning of the compatibility equations (1.3) is that a continuous medium in the unstrained and strained states belongs to a Euclidean space for which $R_{sqtr} \equiv 0$.

The six equations (1.3) were obtained for three dimensions by Saint-Venant (1860). Later, Bussinesq (1871), Beltrami (1889), and Cesaro (1906) [3] proved their sufficiency for simply-connected domains V.

It is obvious that in $\mathbb{R}^n$ Eqs (1.3) may be split into three groups

1. All the free indices s, q, t and r are different (this is the case beginning with $n = 4$). We shall not write down the relations obtained, but just calculate their number N_{n1}. By virtue of the symmetries (1.4) and the Ricci identities (1.5), we have

$$N_{n1} = 3C_n^4 - C_n^4 = n(n-1)(n-2)(n-3)/12$$

2. Only one of the pair of free indices s, q is the same as one of the pair t, r ($n \geq 3$). Then

$$N_{n2} = 3C_n^3 = n(n-1)(n-2)/2$$

3. The pairs of indices s, q and t, r are identical and

$$N_{n3} = b_n = n(n-1)/2$$

Finally, we have

$$N_n = N_{n1} + N_{n2} + N_{n3} = n^2(n^2-1)/12 \tag{1.6}$$

Since the first derivation of Eqs (1.3) [4], repeated discussions have been devoted to the problem of determining the independent equations among all groups and expressing the others in terms of these independent ones [5, 6] (see also the bibliography in [7]). An apparent argument is the fact that N_n for $n > 2$ is greater than the degree of over-determination b_n of the Cauchy system (1.2) ($N_2 = b_2 = 1$). In addition, $N_n \sim n^4$ as n increases, while $b_n \sim n^2$. Appeal has frequently been made to the Bianchi identities

$$R_{sqtr,p} + R_{sqrp,t} + R_{sqpt,r} = 0 \tag{1.7}$$

which are easily rewritten in terms of three derivatives of the components ε_{ij}. For $n = 3$, the number of independent identities (1.7) is just 3, which seems to dispose of the discrepancy $N_3 - b_3 = 3$, and their existence is therefore identified with the dependence of the six strain compatibility equations.

It should be noted that as n increases, the number of Bianchi identities (1.7) increases as $C_n^2C_n^3 \sim n^5$, which even formally does not resolve the discrepancy $N_n - b_n \sim n^4$. At the same time, the identities (1.7) are differential constraints of the third order imposed on ε_{ij}, not additional compatibility equations.

Here is an illustrative example. Let the function $f(x_1, \ldots, x_n)$ satisfy a system of n differential equations

$$\varphi_1 \equiv \partial f/\partial x_1 = 0, \ldots, \varphi_n \equiv \partial f/\partial x_n = 0 \tag{1.8}$$

which is compatible and has the solution $f \equiv \text{const}$. In addition, all Eqs (1.8) are clearly independent (in the sense that deleting just one of them yields a non-equivalent system). The degree of over-determination of the system is $n - 1$. At the same time, $n(n-1)/2$ independent identities $\varphi_{i,j} - \varphi_{j,i} = 0$ exist, which play the part of the identities (1.7), whose existence does not eliminate the discrepancy $n - 1$ and makes none of Eqs (1.8) dependent on the others.

2. FORMULATION OF THE PROBLEM OF THE THEORY OF ELASTICITY IN TERMS OF STRESSES

Closely related to the foregoing discussion are questions of different formulations of the problem of the mechanics of deformable solids in terms of stresses. In what follows we take $n = 3$ and drop the superscripts indicating the ranks of the tensors $\varepsilon^{\{2\}}$, $\sigma^{\{2\}}$ and $\eta^{\{2\}}$.

As is well known [8], the classical formulation of the static problem of the isotropic theory of elasticity in terms of stresses consists of solving, in a three-dimensional domain V, the three equilibrium equations, which are, in vector form

$$\mathbf{S} \equiv \mathrm{Div}\boldsymbol{\sigma} + \rho\mathbf{F} = 0, \quad \mathbf{r} \in V \tag{2.1}$$

and the six Saint-Venant compatibility equations

$$\boldsymbol{\eta}(\boldsymbol{\sigma}) = 0, \quad \mathbf{r} \in V \tag{2.2}$$

where $\boldsymbol{\eta}$ is the Kröner incompatibility tensor with Cartesian components

$$\eta_{ij} = \epsilon_{ikl}\epsilon_{jmp}\varepsilon_{lm,\,kp} \tag{2.3}$$

which follow from Eqs (1.1); the solution is also required to satisfy three static boundary conditions on the boundary $\Sigma = V$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{P}^{(0)}, \quad \mathbf{r} \in \Sigma \tag{2.4}$$

where $\rho\mathbf{F}$ and $\mathbf{P}^{(0)}$ are given volume and surface loads, and $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are the tensors of stresses and small deformations, which are related by Hooke's law

$$\boldsymbol{\varepsilon} = \frac{1}{E}[-3\nu\sigma\mathbf{I} + (1+\nu)\boldsymbol{\sigma}], \quad \sigma = \frac{1}{3}\mathrm{tr}\boldsymbol{\sigma} \tag{2.5}$$

System (2.1), (2.2) is equivalent to the system of three equations (2.1) and the six Beltrami–Mitchell equations, written in tensor form as

$$\mathbf{H} \equiv \Delta\boldsymbol{\sigma} + \frac{3}{1+\nu}\mathrm{Grad}(\mathrm{grad}\sigma) + 2\rho\mathrm{Def}\mathbf{F} + \frac{\rho\nu}{1-\nu}(\mathrm{div}\mathbf{F})\mathbf{I} = 0, \quad \mathbf{r} \in V \tag{2.6}$$

The solution of boundary-value problem (2.1), (2.2), (2.4) (or (2.1), (2.6), (2.4)) is unique if $E > 0$ and $-1 < \nu < 1/2$. The classical formulation (2.1), (2.6), (2.4) corresponds to the variational formulation for the Castigliano functional. This variational principle is the basis of the Filonenko–Borodich method for the approximate solution of the static problem of the theory of elasticity in terms of stresses [9].

Since Mitchell (1900) derived Eqs (2.6), which had in fact been obtained previously by Beltrami (1892) assuming zero volume forces, many investigators have tackled the problem of the over-determination of the classical formulation, since the number of Eqs (2.1) and (2.6) in the domain V is three times the number of unknown components of the tensor $\boldsymbol{\sigma}$. At the same time, the number of boundary conditions (2.4) is only one third of the number of unknown functions. In that connection, it has been proposed that three of the nine equations (2.1), (2.6) (in different combinations) may be removed on the boundary Σ.

These questions are of course in harmony with the question discussed in Section 1 of the number of independent Saint-Venant compatibility equations. The apparent over-determination is removed by a new formulation of the problem of the mechanics of deformable solids in terms of stresses, proposed by one of us [10, 11]. As applied to the isotropic theory of elasticity, it consists of finding the six components of the symmetric tensor $\boldsymbol{\sigma}$ by solving the six equations (2.6) in the domain V, assuming the satisfaction on Σ of the three boundary conditions (2.4) and the three conditions

$$\mathbf{S} = 0, \quad \mathbf{r} \in \Sigma \tag{2.7}$$

This formulation differs from the classical one in that satisfaction of the equilibrium equations is required only on the boundary of the body (at infinity if the body is unbounded). The existence theorem for a solution of the problem of the theory of elasticity in terms of stresses has been proved for the first time, as well as its ellipticity; and a new variational principle has been formulated for a certain scalar operator depending on the stress gradient.

However, attempts have been made in some publications to remove from the nine equations (2.1), (2.6) on the boundary of V not (2.7), but other triples, taken from the Beltrami–Mitchell equations (2.6). The most frequent among these triples are those corresponding to the three diagonal components of the tensor $\mathbf{H}$ or the three non-diagonal ones.

3. SOME COUNTEREXAMPLES

The following examples illustrate that the formulations obtained in that case are not equivalent to the classical formulation of the problem of the theory of elasticity in terms of stresses. To fix our ideas, let us assume that $\mathbf{F} \equiv 0$, choose a spherical system of coordinates $\{r;\ \theta;\ \varphi\}$, and confine our attention to the case in which the stress field is independent of the meridional angle φ. Then

$$S_r = \sigma_{rr,r} + \frac{1}{r}\sigma_{r\theta,\theta} + \frac{1}{r}(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\varphi\varphi} + \sigma_{r\theta}\operatorname{ctg}\theta)$$

$$S_\theta = \sigma_{r\theta,r} + \frac{1}{r}\sigma_{\theta\theta,\theta} + \frac{1}{r}((\sigma_{\theta\theta} - \sigma_{\varphi\varphi})\operatorname{ctg}\theta + 3\sigma_{r\theta})$$

$$S_\varphi = \sigma_{r\varphi,r} + \frac{1}{r}\sigma_{\theta\varphi,\theta} + \frac{1}{r}(3\sigma_{r\varphi} + 2\sigma_{\theta\varphi}\operatorname{ctg}\theta)$$

$$\Delta\sigma_{\alpha\beta} = \frac{1}{r^2}(r^2\sigma_{\alpha\beta,r})_{,r} + \frac{1}{r^2\sin^2\theta}(\sigma_{\alpha\beta,\theta}\sin\theta)_{,\theta}, \quad \alpha, \beta = r, \theta, \varphi$$

Example 1. The problem is to solve the following six equations in the domain of the body

$$S_r = S_\theta = S_\varphi = 0, \quad H_{r\theta} = H_{\theta\varphi} = H_{r\varphi} = 0, \quad \mathbf{r} \in V \tag{3.1}$$

in such a way that six conditions are satisfied on the boundary: the three static conditions (2.4) and the conditions

$$H_{rr} = H_{\theta\theta} = H_{\varphi\varphi} = 0, \quad \mathbf{r} \in \Sigma \tag{3.2}$$

The following stress field is a solution of problem (3.1), (2.1), (3.2) but not of problem (2.1), (2.6), (2.4) (or (2.6), (2.4). (2.7)). We take

$$V = \{\mathbf{r}: r < R\}, \quad \Sigma = \{\mathbf{r}: r = R\} \tag{3.3}$$

$$P_r^0 = 3R^4, \quad P_\theta^0 = P_\varphi^0 = 0, \quad \mathbf{r} \in \Sigma \tag{3.4}$$

and define, for examples

$$\begin{aligned} \sigma_{rr} &= 3r^4 - 10Rr^3 + 10R^2r^2 \\ \sigma_{\theta\theta} &= \sigma_{\varphi\varphi} = 9r^4 - 25Rr^3 + 20R^2r^2, \quad \sigma_{\alpha\beta} \equiv 0 \end{aligned} \tag{3.5}$$

One can verity directly that Eqs (3.1) are satisfied throughout V, and the boundary conditions (2.4) with loads (3.4) are satisfied on the surface of the sphere (3.3). In addition,

$$H_{rr} = 60(r-R)^2 + \frac{60}{1+\nu}(r-R)(7r-5R) \not\equiv 0$$

$$H_{\theta\theta} = H_{\varphi\varphi} = 60(r-R)(3r-2R) \not\equiv 0$$

which means that conditions (3.2) are satisfied but conditions (2.6) are not.

Example 2. The problem is to solve the following six equations in the domain of the body

$$S_r = S_\theta = S_\varphi = 0, \quad H_{rr} = H_{\theta\theta} = H_{\varphi\varphi} = 0, \quad \mathbf{r} \in V \tag{3.6}$$

in such a way that the three static conditions (2.4) and the conditions

$$\begin{aligned} H_{r\theta} &= H_{\theta\varphi} = H_{r\varphi} = 0, \quad \mathbf{r} \in \Sigma \\ H_{r\theta} &= H_{\theta\varphi} = H_{r\varphi} = 0, \quad \mathbf{r} \in V, \quad r \to \infty \end{aligned} \tag{3.7}$$

are satisfied. We take (see Fig. 1)

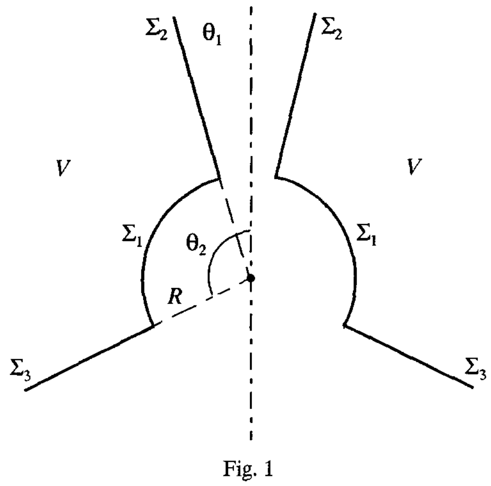

Fig. 1

$$V = \{\mathbf{r}: r > R,\ 0 < \theta_1 < \theta < \theta_2 < \pi\}, \quad \Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$$
$$\Sigma_1 = \{\mathbf{r}: r = R,\ \theta_1 < \theta < \theta_2\} \tag{3.8}$$

$$\Sigma_2 = \{\mathbf{r}: r > R,\ \theta = \theta_1\}, \quad \Sigma_3 = \{\mathbf{r}: r > R,\ \theta = \theta_2\}$$
$$P_r^0 = P_\theta^0 = P_\varphi^0 = 0, \quad \mathbf{r} \in \Sigma \tag{3.9}$$

and present a stress field which solves problem (3.6), (2.4), (3.7) but not problem (2.1), (2.6), (2.4).

Put

$$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{\varphi\varphi} = \sigma_{r\theta} \equiv 0$$
$$\sigma_{r\varphi} = \frac{(r-R)^3}{r^3 \sin^2\theta} f', \quad \sigma_{\theta\varphi} = -\frac{3(r-R)^2}{r^2 \sin^2\theta} f \tag{3.10}$$

where $f(\theta)$ is some differentiable function, thereby satisfying Eqs (3.6). To satisfy conditions (3.7) and (3.9) on the surface Σ and at infinity, it is sufficient to take, for example,

$$f(\theta) = (\theta - \theta_1)^4 (\theta - \theta_2)^4 \tag{3.11}$$

The explicit expressions for $H_{r\varphi} = \Delta\sigma_{r\varphi}$ and $H_{\theta\varphi} = \Delta\sigma_{\theta\varphi}$ are too cumbersome to present here. We note that the components $H_{r\varphi}$ and $H_{\theta\varphi}$ evaluated from (3.11) are bounded in the domain V, since the whole polar axis $\sin\theta = 0$ does not belong to V, and they vanish on Σ_1, Σ_2, Σ_3 and at infinity. The components $H_{r\varphi}$ and $H_{\theta\varphi}$ of the tensor (2.6) do not vanish identically in the domain V.

Thus, removal of the three "diagonal" or "non-diagonal" Beltrami–Mitchell equations on the boundary makes the formulation of the problem in terms of stresses non-equivalent to the classical formulation. Counterexamples can be presented analogously which in similar fashion illustrate the inadmissibility of removing all other triples of the nine equations (2.1), (2.6) on the boundary (except for the triple (2.1)).

This research was supported financially by the Russian Foundation for Basic Research (02-01-00780).

REFERENCES

1. VOLTERRA, V., Sur l'équilibre des corps élastiques multiplement connexes. *Ann. l'École Norm. Sup.*, 1907, **24**, 401–507.
2. KRÖNER, E., *Kontinuumtheorie der Versetzungen und Eigenspannungen*. Springer, Berlin, 1958.
3. CESARO, E., Sulle formole del Volterra, fondamentali nella teoria delle distorzioni elastiche. *Rend. Acad. R. Napoli*, 1906, **12**, 311–321.
4. BELTRAMI, E., Osservazioni sulla nota precedente. *Atti Accad. Lincei R. C.*, 1892, **1**, 141–142.
5. WASHIZU, K., *Variational Methods in Elasticity and Plasticity*. Pergamon Press, Oxford, 1982.

6. HAHN, H.G., *Elastizitätstheorie*. Teubner, Stuttgart, 1985.
7. POBEDRYA, B. Ye., The static problem in terms of stresses. Vestnik MGU, *Ser. 1, Matematika, Mekhanika*, 2003, **3**, 61–67.
8. NOWACKI, W., *Theory of Elasticity*. PWN, Warsaw, 1970.
9. FILONENKO-BORODICH, M. M., *Theory of Elasticity*. Fizmatgiz, Moscow, 1959.
10. POBEDRYA, B. Ye., Anew formulation of the problem of the mechanics of a deformable solid in terms of stresses. *Dokl. Akad. Nauk SSSR*, 1980, **253**, 2, 295–297.
11. POBEDRYA, B. Ye., SHESHENIN, S. V. and KHOLMATOV, T., *Problem in Stresses*. Fan, Tashkent, 1988.

Translated by D.L.